

A new approach for analyzing panel AR(1) series with application to the unit root test

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SUMMARY

This paper derives several novel tests to improve on the t-test for testing AR(1) coefficients of panel time series, i.e., of multiple time series, when each has a small number of observations. These tests can determine the acceptance or the rejection of each hypothesis individually while controlling the average type one error. Strikingly, the testing statistics derived by the empirical Bayes approach can be approximated by a simple form similar to the t-statistic; the only difference is that the means and the variances are estimated by shrinkage estimators. Simulations demonstrate that the proposed tests have higher average power than the t-test in all settings we examine including those when the priors are miss-specified and the cross section series are dependent.

JEL Classification: C12; C32.

Keywords: Empirical Bayes; Multiple tests; Panel time series; Random effect model; Shrinkage estimator.

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1 Introduction

Testing a unit root hypothesis is a very important subject since, for example, it can be applied to test the Purchasing Power Parity (PPP) theory in Economics. The topic attracts much attention in research also because the traditional unit root tests can have low power in some circumstances. One such circumstance relates to the panel time series which consists of N series to be simultaneously tested, each having T observations, where N is large or moderate and T is small. This is the scenario we consider in this paper. Although we focus on the Economics settings when discussing applications, our solutions are applicable to other applications involving the “small n and large p ” problems in Statistics.

In this paper, we consider a new approach to derive several novel tests that improve, in power, on the traditional t-test for testing multiple unit root null hypotheses and white noise null hypotheses. This new approach is based on the optimal multiple test criterion (Liu, 2006, Storey, 2007, Storey et al., 2007, Hwang and Liu, 2010, and Noma and Matsui, 2012 and 2013), which are motivated by biological microarray data analyses. This approach could determine acceptance or rejection of each hypothesis individually while controlling the average type one error of the multiple tests. Traditionally, the panel unit root approach tests against the null hypothesis that all series follow a unit root model (Levin et al., 1992, Baltagi and Kao, 2000, Bai and Ng, 2004, 2010, Pesaran, 2007, Pesaran, et al., 2013, and etc.). Hence either all series are declared a unit root model (i.e. non-stationary series) or some declared stationary series without identifying which. In contrast, the tests proposed in this paper could determine the stationarity of each series and in the meantime control the average type one error. This seems more desirable.

The multiple test criterion considered by Liu (2006), Storey (2007), Hwang and Liu (2010), and Noma and Matsui (2012, 2013), are to maximize the average power while controlling the average type one error. Interestingly, such a criterion is equivalent to other optimality criteria based on controlling the false discovery rate (FDR). See Storey (2007) and Hwang and Liu (2010). Their approaches and the approach in this paper all use the Neymann-Pearson fundamental lemma to derive optimum procedures.

While all aim at controlling the average Frequentist type one errors, the difference between the approaches of Storey (2007), and the group of researchers, Hwang and

Liu (2010) and Noma and Matusi (2012, 2013) is that Storey’s approaches aims to maximize the Frequentist average powers whereas the others aim to maximize the Bayesian average powers. The advantage of the approaches of Hwang and Liu (2010) and Noma and Matusi (2012, 2013) over Storey’s approach is that the former are much faster in computation and also, as shown in Hwang and Liu (2010), provide higher average test power. While the approaches of Hwang and Liu (2010) and Noma and Matusi (2012, 2013) are similar, the statistics proposed by Hwang and Liu have further approximated formulae in simpler forms, which can be easily calculated without evaluating integral, unlike Norma and Matsui’s approach which requires evaluating $3N$ integrals.

In this paper, we tackle the difficult problem of testing coefficients of time series models. We follow the approach of Hwang and Liu (2010) which constructs the MAP test, i.e., the test that maximizes the Bayesian expected average power with respect to a prior distribution while controlling the Frequentist average type one error. The general theory developed in Section 3 shows that the MAP test is an approximation of Story’s test. To derive the statistics for testing the one-sided and two-sided hypotheses of the coefficients of panel AR(1) models, we assume a class of priors on the means, a class of priors on the variances, or on both resulting in a MAP statistic shrinking the means, the variances or both respectively. These statistics are further approximated, leading to the proposed statistics. Strikingly, in all situations, the proposed statistics basically take a simple form similar to the t-statistic; the only difference is that the means and the variances are estimated by shrinkage estimators. Previously, such a result was available in Hwang and Liu (2010), Cui et al. (2005), and Smyth (2004) only for the usual ANOVA models and only for the procedure shrinking the variances. For the procedure shrinking the means and the variances, the tests of Hwang and Liu (2010) were not put in the form of the t-statistic.

Our proposed shrinkage t-tests are shown to have higher average powers than the traditional t-test because the tests “borrow the strength” from all series. The tests implicitly determine how similar the parameters of the series are. The more similar the series are, the more extensively the data from other cross sections are used to estimate the parameters of the individual series. Consequently, the improvements are larger.

Note that the testing statistics developed in this paper aim to satisfy the Frequentist criteria of controlling the average type one error, even though we use a Bayesian

approach to construct the proposed statistics. To be more realistic, we consider not only a prior but a class of priors indexed by some hyper-parameters. We use the data to estimate the hyper-parameters; hence the procedure is called empirical Bayes, which is equivalent to the Frequentist approach based on a random effect model. Hence our results are quite different from the Bayesian unit root tests proposed in Uhlig (1994) and Philips and Xiao (1998).

This paper uses a bootstrap method to obtain the critical value to control the average type one error of the proposed tests. Simulations in Section 7 show that bootstrap works well for all our settings. Note that our problem is different from the unit root bootstrap tests proposed by Ferretti and Romo (1996) and Park (2003), which aim at testing a single hypothesis with a large number, T , of observations. Simulation results also show that the proposed tests have either higher or similar average power when compared with the t-tests in all the cases we considered. Specifically, when $N = 80$ and $T = 10$, the proposed tests increase the average power of t-test by 70% and 25%, respectively, for testing the white noise null hypotheses and the unit root null hypotheses. We also demonstrate similar improvement when the model is misspecified and when the cross section series are dependent. In this paper, although we only work on AR(1) models, we anticipate that these results can be generalized to more complex time series models.

Our proposed tests are fast in computation. Given a data set with $N = 1000$ and $T = 10$, it takes about 10 seconds to compute our proposed tests (F_{SS} and RF_{SS}) for all 1000 hypotheses, using a Laptop and the GAUSS 9.0 program.

The rest of this paper is organized as follows. In Section 2, we present the model considered in this research and give a review of the optimal discovery procedure (Storey, 2007) and the maximizing average power test (MAP) (Hwang and Liu, 2010). In Section 3, we develop a theory that links the two approaches. In Sections 4 and 5, we derive the MAP test under various prior assumptions for the two-sided and the one-sided hypotheses. The proposed empirical Bayes tests are constructed in Section 6, where the issues of estimating the hyper-parameters and controlling the average type one error by bootstrap are discussed. In Section 7, we present the simulation results. Section 8 gives the concluding remarks.

2 The Model and Reviews of the ODP and MAP Tests

Suppose the N -dimensional AR(1) processes are generated by

$$y_{j,t} = \phi_j y_{j,t-1} + e_{j,t} \quad \text{for } 1 \leq j \leq N \text{ and } 1 \leq t \leq T, \quad (1)$$

where for section j , $e_{j,t}$ is an i.i.d. normal random variable with zero mean and variance σ_j^2 . Note that we allow the dependence of the cross section series in model (1). Except in Sections 6 and 7, we derive the tests without assuming that the cross section series are independent throughout the paper.

The observation of the j -th section is $\mathbf{y}_j = (y_{j,1}, \dots, y_{j,T})'$, and the probability density function (pdf) of \mathbf{y}_j given $y_{j,1}$ is

$$\left(\frac{1}{\sqrt{2\pi}\sigma_j}\right)^{T-1} e^{-\frac{1}{2}\sum_{t=2}^T (y_{j,t} - \phi_j y_{j,t-1})^2 / \sigma_j^2}. \quad (2)$$

It is easy to see that $\sum_{t=2}^T (y_{j,t} - \phi_j y_{j,t-1})^2 = \sum_{t=2}^T (y_{j,t} - \hat{\phi}_j y_{j,t-1})^2 + (\hat{\phi}_j - \phi_j)^2 \sum_{t=2}^T y_{j,t-1}^2$, where

$$\hat{\phi}_j = \frac{\sum_{t=2}^T y_{j,t} y_{j,t-1}}{\sum_{t=2}^T y_{j,t-1}^2} \quad (3)$$

is the least squares estimator by regressing $y_{j,t}$'s against $y_{j,t-1}$'s. Using the notation

$$\hat{\sigma}_j^2 = (1/(T-1)) \sum_{t=2}^T (y_{j,t} - \hat{\phi}_j y_{j,t-1})^2 \quad \text{and} \quad S_j = \sum_{t=2}^T y_{j,t-1}^2, \quad (4)$$

the pdf of \mathbf{y}_j can be written as

$$\left(\frac{1}{\sqrt{2\pi}\sigma_j}\right)^{T-1} e^{-\frac{1}{2}(T-1)\hat{\sigma}_j^2/\sigma_j^2} \cdot e^{-\frac{1}{2}(\hat{\phi}_j - \phi_j)^2 S_j / \sigma_j^2}. \quad (5)$$

It follows that $(\hat{\phi}_j, \hat{\sigma}_j^2, S_j)$ is a sufficient statistic. Also $(\hat{\phi}_j, \hat{\sigma}_j^2)$ is the maximum likelihood estimator of (ϕ_j, σ_j^2) .

In this paper, we consider testing simultaneously the N hypotheses

$$H_0^j : \phi_j = \phi_0 \quad \text{vs.} \quad H_1^j : \phi_j \in D, \quad (6)$$

where ϕ_0 is a fixed number, $j = 1, \dots, N$, and D denotes a set of ϕ_j 's. When D consists of a single point, (6) corresponds to a simple test. When $D = \{\phi | \phi \neq \phi_0\}$,

(6) corresponds to a two-sided test and when $D = \{\phi | \phi \leq \phi_0\}$, (6) corresponds to a one-sided test.

The optimal discovery procedure (ODP) of Storey (2007) aims at constructing a rejection region that maximizes the expected number of the true positive results (ETP) while controlling the expected number of false positive results (EFP). The discussion below is applicable to the general situation, where \mathbf{y}_j , a T -dimensional random vector, has pdf $f(\mathbf{y}_j | \phi_j, \sigma_j)$. The procedures we consider are to

$$\text{reject } H_0^j \quad \text{if } \mathbf{y}_j \in C, \quad (7)$$

where C , independent of j , is a set of T -dimensional vectors. Storey (2007) argues that procedures using C depending on j have no advantage in average power. Using $I(\cdot)$ to denote an indicator function, we have

$$\text{ETP} = E\left\{\sum_{j=1}^N I(\phi_j \in D \text{ and } \mathbf{y}_j \in C)\right\} = \sum_{\{j | \phi_j \in D\}} P_{\phi_j, \sigma_j}(\mathbf{y}_j \in C) = \int_{y \in C} \sum_{\{j | \phi_j \in D\}} f(\mathbf{y} | \phi_j, \sigma_j) d\mathbf{y}. \quad (8)$$

Similarly,

$$\text{EFP} = \sum_{\{j | \phi_j = \phi_0\}} P_{\phi_j, \sigma_j}(\mathbf{y}_j \in C) = \int_{y \in C} \sum_{\{j | \phi_j = \phi_0\}} f(\mathbf{y} | \phi_j, \sigma_j) d\mathbf{y}. \quad (9)$$

Let N_1 and $N - N_1$ be the number of series satisfying the alternative and the null hypotheses, respectively. Note that ETP/N_1 and $\text{EFP}/(N - N_1)$ are therefore the average power and the average type one error, which are quantities of great concern to a Frequentist.

Applying Neymann-Pearson lemma to (8) and (9) leads to the optimal test which maximizes ETP (or equivalently ETP/N_1) while controlling $\text{EFP}/(N - N_1)$ at level α . Assume that the pdf of \mathbf{y}_j is $f(\mathbf{y}_j | \phi_j, \sigma_j)$, where ϕ_j 's are the key parameters whereas σ_j 's are the nuisance parameters, which can be interpreted as variances. See, for example, the pdf in (2). The rejection region is then $\mathbf{y}_j \in C$, where

$$C = \{\mathbf{y} | \frac{\sum_{\{j | \phi_j \in D\}} f(\mathbf{y} | \phi_j, \sigma_j)}{\sum_{\{j | \phi_j = \phi_0\}} f(\mathbf{y} | \phi_j, \sigma_j)} > \text{crit}\}, \quad (10)$$

where crit is a cutoff point chosen so that it has average type one error equal to α .

There is, however, a problem with the "test" in (10). In order to apply (10), one needs to know which hypothesis is true and which is false, the very information

that one is trying to determine. When σ_j are all equal (to σ), there is however a possible way out, as described in Storey (2007). Write the inequality in (10) as $A/D > \text{crit}$ where A and D represent the numerator and the denominator of sums of probability density functions. Notice that the inequality in (10) is equivalent to $(A + D)/D > \text{crit} + 1$. Also D equals $(N - N_1) \cdot f(\mathbf{y}|\phi_0, \sigma)$. Putting all these together and omitting some constants, the statistic is equivalent to $\sum_{j=1}^N f(\mathbf{y}|\phi_j, \sigma)/f(\mathbf{y}|\phi_0, \sigma)$, which can be calculated without knowing which hypothesis is true or false.

However, when σ_j 's are different, it is much harder to approximate the statistic in (10). Storey, et al. (2007) did have a successful attempt, where σ_j is replaced by an estimator based on the j th population. Storey's procedure, however, is computationally very intensive. It requires calculating N times the statistic, which involves summation of N terms. When N is large, it is overwhelming.

The approach of Hwang and Liu (2010) is more parametric because of postulating classes of prior distributions on both ϕ_j and σ_j . They construct their MAP (acronym of maximum average power) test to maximize the Bayesian expected value of ETP, which is the average power with respect to the prior distribution. Using some intuitive approximation in the empirical Bayes fashion, they were able to derive some statistics which can be calculated instantaneously. In particular, their approach leads to a statistic, called F_{SS} , not only borrowing the strength from all populations to estimate ϕ_j (which Storey's procedure does), but also to estimate σ_j (which Storey's procedure does not). Consequently, it is to be expected that F_{SS} test has higher average power, which was numerically demonstrated to be so.

In this paper, the approach of Hwang and Liu (2010) is applied to the time series models which are much more difficult to construct statistical tests than their ANOVA models. To the best of our knowledge, this paper is the first to present shrinkage multiple tests for the time series model. The empirical Bayes approach is equivalent to the random effect model approach, because the parameters are assumed to be random in either case.

3 The Main Theorems

In this section, we shall provide a general theory that shows that the ODP approach of Storey (2007) is asymptotically equivalent to the MAP test of Hwang and Liu (2010). We then apply the theory to the AR(1) model in the following sections. In

the case when a theorem needs a proof, the proof is provided in the Appendix. In this section, we consider testing the hypothesis (6) by assuming that each \mathbf{y}_j has the pdf $f(\mathbf{y}_j|\phi_j, \sigma_j)$. Under the assumption, we consider three cases of priors:

Case 1: ϕ_j is fixed and σ_j 's are i.i.d. each having the distribution $\pi_1(\sigma)$.

Case 2: σ_j 's are fixed and ϕ_j are i.i.d., each having the distribution $\pi_2(\phi)$.

Case 3: (ϕ_j, σ_j) are i.i.d., each having the distribution $\pi(\phi, \sigma)$.

Considerations of these three cases shall, in order, lead to F_{sv} , F_{sm} and F_{ss} tests for the two-sided hypothesis and to RF_{sv} , RF_{sm} and RF_{ss} for the one-sided hypothesis. The subscripts sv, sm, and ss, represent tests that shrink the variances, shrink the means and shrink both the means and variances, respectively.

We first write an asymptotic formula for a rejection region C for Case 1, which rejects H_0^j if and only if $\mathbf{y}_j \in C$.

Theorem 1 (Case 1). Under the assumption of Case 1, as N_1 and $N - N_1$ go to infinity,

$$\frac{\text{ETP}}{N_1} - \text{BETP} \longrightarrow 0 \quad \text{in probability} \quad (11)$$

and

$$\frac{\text{EFP}}{N - N_1} - \text{BEFP} \longrightarrow 0 \quad \text{in probability}, \quad (12)$$

where

$$\text{BETP} = \frac{1}{N_1} \sum_{\{j|\phi_j \in D\}} \int P_{\phi_j, \sigma}(\mathbf{y} \in C) d\pi_1(\sigma),$$

and

$$\text{BEFP} = \int P_{\phi_0, \sigma}(\mathbf{y} \in C) d\pi_1(\sigma).$$

The “B” in the notation of BETP and BEFP stands for “Bayes”. Actually, both quantities are also the Bayesian expected values of the Frequentist’s quantities, ETP/N_1 and $\text{EFP}/(N - N_1)$. The MAP test of Hwang and Liu (2010) is defined as the test that maximizes the Bayesian expectation of the average power, BETP, among all tests such that

$$\text{BEFP} \leq \alpha. \quad (13)$$

Hence the theorem shows that the ODP approach of Storey (2007) is asymptotically equivalent to the MAP test of Hwang and Liu (2010).

The proof of the above theorem using the law of large numbers is based on the assumption that σ'_j s are independent. However, even if σ'_j s are correlated, it is possible to write weaker conditions so that the law of large numbers applies and hence the theorem could be established under weaker assumptions.

Next, by interchanging the order of integration, we can write BETP and BEFP as

$$\text{BETP} = \int_{\mathbf{y} \in C} \frac{1}{N_1} \sum_{\{j|\phi_j \in D\}} \int f(\mathbf{y}|\phi_j, \sigma) d\pi_1(\sigma) d\mathbf{y}$$

and

$$\text{BEFP} = \int_{\mathbf{y} \in C} \int f(\mathbf{y}|\phi_0, \sigma) d\pi_1(\sigma) d\mathbf{y}.$$

Therefore the Neymann-Pearson fundamental lemma implies the following theorem.

Theorem 2 (Case 1). Among all the procedures in (7), the MAP test consists of \mathbf{y} such that

$$\text{PT}(\phi_1, \dots, \phi_N) \equiv \frac{\sum_{\{j|\phi_j \in D\}} \int f(\mathbf{y}|\phi_j, \sigma) d\pi_1(\sigma)}{\int f(\mathbf{y}|\phi_0, \sigma) d\pi_1(\sigma)} > \text{crit}, \quad (14)$$

where crit is chosen so that equality in (13) is attained for the test (14).

There is, however, a problem with PT in (14), which stands for “pseudo test”. Namely, it still depends on the unknown parameters ϕ'_j s and is not really an applicable test. Following the principle of likelihood ratio test, we can use the statistic

$$\sup_{\phi_1, \dots, \phi_N \in D} \text{PT}(\phi_1, \dots, \phi_N), \quad (15)$$

which leads to a real statistical test. Note we can view (15) as an approximate MAP test, since the ϕ'_j s are replaced by the maximum likelihood estimators $\hat{\phi}'_j$ s. The approximation is one of the best imaginable approximations, even when the sample sizes are small.

We now state the theorem which, under a condition, gives us an explicit formula for (15).

Theorem 3 (Case 1). Assume that the maximization of $f(\mathbf{y}|\phi, \sigma)$ with respect to $\phi \in D$ is attained when $\phi = \hat{\phi}_M(\mathbf{y}) \in D$ and $\hat{\phi}_M$ does not depend on σ_j . Then (15) equals

$$\frac{\int f(\mathbf{y}|\hat{\phi}_M, \sigma) d\pi_1(\sigma)}{\int f(\mathbf{y}|\phi_0, \sigma) d\pi_1(\sigma)}. \quad (16)$$

For Case 2, similar to Theorems 1 and 2, we could establish results which are stated below while omitting the proof.

Theorem 4 (Case 2). Under the assumption of Case 2, the statement in Theorem 1 holds with

$$\text{BETP} = \frac{1}{N_1} \sum_{\{j|H_0^j \text{ is false}\}} \int P_{\phi, \sigma_j}(\mathbf{y} \in C) d\pi_2(\phi)$$

and

$$\text{BEFP} = \frac{1}{N - N_1} \sum_{\{j|H_0^j \text{ is true}\}} P_{\phi_0, \sigma_j}(\mathbf{y} \in C).$$

Also the MAP test for this case rejects H_0^j if and only if $\mathbf{y}_j \in C$ and

$$C = \{\mathbf{y} | \frac{\sum_{\{j|H_0^j \text{ is false}\}} \int f(\mathbf{y}|\phi, \sigma_j) d\pi_2(\phi)}{\sum_{\{j|H_0^j \text{ is true}\}} \int f(\mathbf{y}|\phi_0, \sigma_j)} > \text{crit}\}, \quad (17)$$

where crit is chosen so that the equality in (13) holds.

The above region is not a usable rejection region, since it depends on unknown σ'_j s. To derive a useful version, we could use the likelihood ratio principle by taking the sup of the numerator and denominator of the ratio in (17). It is easy to see that the resultant ratio is proportional to

$$\frac{\int f(\mathbf{y}|\phi, \hat{\sigma}_M) d\pi_2(\phi)}{f(\mathbf{y}|\phi_0, \hat{\sigma}_0)}, \quad (18)$$

where $\hat{\sigma}_M$ maximizes $\int f(\mathbf{y}|\phi, \sigma) d\pi_2(\phi)$ and $\hat{\sigma}_0$ maximizes $\int f(\mathbf{y}|\phi_0, \sigma) d\pi_2(\phi)$, since $\hat{\sigma}_M$ and $\hat{\sigma}_0$ do not depend on j . Now (18) is a real statistic, since it does not depend on σ'_j s. We note here that the process of turning (17) into a real statistic can also be carried out with estimators which are different from the maximizers. Later on, it turns out that the close form of $\hat{\sigma}_M$ can not be easily derived for the AR(1) model and so we use a different estimator $\hat{\sigma}_\star$ to substitute for σ_j in (17). This leads to (18) with $\hat{\sigma}_M$ being replaced by $\hat{\sigma}_\star$ i.e.

$$\frac{\int f(\mathbf{y}|\phi, \hat{\sigma}_\star) d\pi_2(\phi)}{f(\mathbf{y}|\phi_0, \hat{\sigma}_0)}. \quad (19)$$

We finally came to the easiest case, Case 3. We state the following theorem and omit the proof, which is similarly to that of Theorem 1.

Theorem 5 (Case 3). Under the assumption of Case 3, the statement in Theorem 1 holds with

$$\text{BETP} = \int P_{\phi, \sigma}(\mathbf{y} \in C) d\pi(\phi, \sigma) = \int_{\mathbf{y} \in C} \int f(\mathbf{y}|\phi, \sigma) d\pi(\phi, \sigma) d\mathbf{y}$$

and

$$\text{BEFP} = \int P_{\phi_0, \sigma}(\mathbf{y} \in C) d\pi_1(\sigma) = \int_{\mathbf{y} \in C} \int f(\mathbf{y}|\phi_0, \sigma) d\pi_1(\sigma) d\mathbf{y}.$$

Consequently, the MAP test rejects H_0^j if $\mathbf{y}_j \in C$ and

$$C = \{\mathbf{y} \mid \frac{\int_{\phi \in D} f(\mathbf{y}|\sigma, \phi) d\pi(\phi, \sigma)}{\int f(\mathbf{y}|\sigma, \phi_0) d\pi_1(\sigma)} > \text{crit}\}, \quad (20)$$

where crit is chosen so that the equality in (13) holds.

The likelihood ratio principle is not used here in deriving Theorem 5.

4 The Two-sided Test

4.1 The t-test

To begin, we consider the two-sided tests

$$H_0^j : \phi_j = \phi_0 \quad \text{vs.} \quad H_1^j : \phi_j \neq \phi_0, \quad \text{where } j = 1, \dots, N. \quad (21)$$

The well-known t-test is to reject H_0^j if t_j^2 is larger than a critical value, where

$$t_j = (\hat{\phi}_j - \phi_0) \left(\frac{S_j}{\hat{\sigma}_j^2} \right)^{1/2}. \quad (22)$$

This test is asymptotically optimal in power if we consider each hypothesis separately. However, tests with larger average power can be constructed as outlined in Section 3. Similar to Hwang and Liu (2010), we construct F_{sv} , F_{sm} , and F_{ss} , which shrink the variances (sv), the means (sm) and both the variances and the means (ss), respectively.

4.2 The test shrinking the variances: F_{SV}

To shrink only the variances and not the means, we shall consider Case 1 in Section 3. The pdf $f(\mathbf{y}_j|\phi_j, \sigma_j)$ is given in (5) and $D = \{\phi : \phi \neq \phi_0\}$. From (5), note that the condition in Theorem 3 is satisfied with $\hat{\phi}_M = \hat{\phi}$, where $\hat{\phi}$ is defined in (3) unless $\hat{\phi}_M = \phi_0$. The latter situation occurs with zero probability and hence can be ignored. It may be easier for the future user to have formulas with \mathbf{y} replaced with \mathbf{y}_j , which we will do. The statistic can then be used directly to determine whether to reject H_0^j . After substituting \mathbf{y} by \mathbf{y}_j , the approximate MAP statistic (16) is identical to

$$\frac{\int (\frac{1}{\sigma})^{T-1} e^{-\frac{(T-1)\hat{\sigma}_j^2}{2\sigma^2}} d\pi_1(\sigma)}{\int (\frac{1}{\sigma})^{T-1} e^{-\frac{(T-1)\hat{\sigma}_j^2}{2\sigma^2} - \frac{S_j(\hat{\phi}_j - \phi_0)^2}{2\sigma^2}} d\pi_1(\sigma)}. \quad (23)$$

Under the assumption that σ^2 has a log-normal distribution with mean μ_V and variance τ_V^2 as in Hwang and Liu (2010), we further approximate (23) by substituting σ by its Bayes estimator; more precisely, $\ln(\sigma^2)$ is substituted by $E(\ln(\sigma^2)|\text{data})$. After estimating μ_V and τ_V^2 by data in the empirical Bayes fashion, we end up with the rejection region

$$F_{SV}^j = t_{jE}^2 > \text{crit}, \quad \text{where} \quad t_{jE} = (\hat{\phi}_j - \phi_0) \left(\frac{S_j}{\hat{\sigma}_{jE}^2} \right)^{1/2}, \quad (24)$$

and σ_{jE}^2 shall be defined below. We note that the critical value, crit, shall be determined using a bootstrap method so its average Frequentist's type one error is bounded by α . The method is applied to all the other tests considered in this paper. We do it this way so the proposed tests have Frequentist's validity.

Note that t_{jE} is the same as t_j with the exception that $\hat{\sigma}_j^2$ in (22) is replaced by $\hat{\sigma}_{jE}^2$, which was proposed by Cui et al. (2005). To define $\hat{\sigma}_{jE}^2$, we take the logarithmic transformation of $\hat{\sigma}_j^2$ and apply the Lindley-James-Stein estimator to estimate $\ln(\sigma_j^2)$. See Lindley (1962) and James and Stein (1961). We then use the exponential Lindley-James-Stein estimator to estimate σ_j^2 . Let $X_j = \ln(\hat{\sigma}_j^2) - E(\ln(\chi_{T-2}^2/T - 2))$, where χ_{T-2}^2 is the Chi-squared random variable with degree of freedom $T - 2$. Hence $\hat{\sigma}_{jE}^2$ is the exponential Lindley-James-Stein estimator, i.e.

$$\hat{\sigma}_{jE}^2 = e^{\delta_{LJS}^j}, \quad (25)$$

where

$$\delta_{LJS}^j = \bar{X} + \left(1 - \frac{(N-3)V_{T-2}}{\sum_{j=1}^N (X_j - \bar{X})^2}\right)_+ (X_j - \bar{X})$$

is the Lindley-James-Stein estimator and V_{T-2} is the variance of $\ln(\chi_{T-2}^2/(T-2))$.

Numerical studies in Section 6 show that (24) has a larger average power than the t-test. To explain this, note that when $\hat{\sigma}_j^2$ are very different from each other, $\sum_{j=1}^N (X_j - \bar{X})^2$ is large. Consequently, δ_{LJS}^j is close to X_j and hence $\hat{\sigma}_{jE}^2$ is close to $\hat{\sigma}_j^2$ up to a constant. Therefore (24) behaves like the t-test and can not be worse. On the other hand, if $\hat{\sigma}_j^2$ are close to each other, resulting in a small $\sum_{j=1}^N (X_j - \bar{X})^2$, δ_{LJS}^j are close to \bar{X} and $\hat{\sigma}_{jE}^2$ are close to the geometric mean of $\hat{\sigma}_j^2$. Since σ_j^2 are likely similar to each other, the geometric mean should be a better estimator than $\hat{\sigma}_j^2$. Hence test (24) is expected to have a larger average power than the t-test, which is confirmed by numerical results.

The Lindley-James-Stein estimator can be derived nonparametrically. Hence it is anticipated that the test of Cui et al. (2005) is robust with respect to the misspecification of the distribution of $\ln(\sigma_j^2)$. The conjecture is supported by the numerical study therein.

4.3 The test shrinking the means: F_{sm}

Now we consider the test that shrinks the means only. Case 2 in Section 3 is assumed and hence Theorem 4 is applicable.

To apply (19) to the AR(1) model, we consider a normal prior:

$$\phi_j \sim N(\mu, \tau^2) \quad \text{when } H_1^j \text{ is true.} \quad (26)$$

Now we shall evaluate the denominator and the numerator of (19) where $f(\cdot|\cdot)$ is defined in (2). The denominator of (19), with \mathbf{y} being replaced by \mathbf{y}_j , equals

$$\sup_{\sigma} f(\mathbf{y}_j | \phi_0, \sigma) = \sup_{\sigma} \left(\frac{1}{\sqrt{2\pi}\sigma} \right)^{T-1} e^{-\frac{(T-1)\hat{\sigma}_{0j}^2}{2\sigma^2}}, \quad (27)$$

where

$$\hat{\sigma}_{0j}^2 = (1/(T-1)) \sum_{t=2}^T (y_{j,t} - \phi_0 y_{j,t-1})^2. \quad (28)$$

Direct calculation shows that maximum occurs at $\sigma^2 = \hat{\sigma}_{0j}^2$. Plugging this into (27) shows that (27) equals

$$\left(\frac{1}{\sqrt{2\pi}\hat{\sigma}_{0j}} \right)^{T-1} e^{-\frac{(T-1)}{2}}. \quad (29)$$

Now to calculate the numerator of (19), we first calculate $\int f(\mathbf{y}_j|\phi, \sigma) d\pi(\phi)$ which can be shown after some direct calculations to equal

$$\left[\left(\frac{1}{\sqrt{2\pi}\sigma_j} \right)^{T-1} e^{-\frac{(T-1)\hat{\sigma}_j^2}{2\sigma_j^2}} \right] \cdot \sqrt{\frac{\sigma_j^2}{S_j\tau^2 + \sigma_j^2}} e^{-\frac{1}{2}(\hat{\phi}_j - \mu)^2 \frac{S_j}{S_j\tau^2 + \sigma_j^2}}. \quad (30)$$

The last expression can be derived using (5), rewriting the second exponential term in (5) as $\frac{\sigma_j}{\sqrt{S_j}} \frac{\sqrt{S_j}}{\sigma_j} e^{-\frac{1}{2}(\hat{\phi}_j - \phi_j)^2 \frac{S_j}{\sigma_j^2}}$ and using the classical Bayesian theory which implies that $\hat{\theta}$ has a $N(\mu, \sigma^2 + \tau^2)$ distribution if $\hat{\theta}|\theta \sim N(\theta, \sigma^2)$ and $\theta \sim N(\mu, \tau^2)$.

Now it seems difficult to find the maximum likelihood estimator of σ_j^2 . Hence instead we use $\hat{\sigma}_j^2$ (defined in (4)), which maximizes the bracket in (30). Hence when $\hat{\sigma}_1$ is taken to be $\hat{\sigma}_j$, (19) now equals the ratio of (30) to (29), which yields

$$(\hat{\sigma}_{0j}^2/\hat{\sigma}_j^2)^{(T-1)/2} \sqrt{\frac{\hat{\sigma}_j^2}{S_j\tau^2 + \hat{\sigma}_j^2}} e^{-\frac{1}{2}(\hat{\phi}_j - \mu)^2 \frac{S_j}{S_j\tau^2 + \hat{\sigma}_j^2}}. \quad (31)$$

To gain some insight about how (31) works, we show that it can be approximated by a formula similar to the t-test. Using the approximation $(\hat{\sigma}_{0j}^2/\hat{\sigma}_j^2)^{(T-1)/2} = (1 + (\hat{\phi}_j - \phi_0)^2 \frac{S_j}{(T-1)\hat{\sigma}_j^2})^{(T-1)/2} \doteq e^{\frac{S_j}{2\hat{\sigma}_j^2}(\hat{\phi}_j - \phi_0)^2}$ when T is large, we express (31) as

$$\sqrt{\frac{\hat{\sigma}_j^2}{S_j\tau^2 + \hat{\sigma}_j^2}} e^{\frac{S_j}{2\hat{\sigma}_j^2}(\hat{\phi}_j - \phi_0)^2 - \frac{1}{2}(\hat{\phi}_j - \mu)^2 \frac{S_j}{S_j\tau^2 + \hat{\sigma}_j^2}}. \quad (32)$$

According to our numerical study, the more important factor in (32) is the exponential part and not the square root factor. Omitting the square root factor and taking a log transformation of the reminder of (32) lead to

$$\begin{aligned} & \frac{S_j}{\hat{\sigma}_j^2}(\hat{\phi}_j - \phi_0)^2 - (\hat{\phi}_j - \mu)^2 \frac{S_j}{S_j\tau^2 + \hat{\sigma}_j^2} \\ &= \left(\frac{\hat{\sigma}_j^2\tau^2}{\tau^2 S_j + \hat{\sigma}_j^2} \right)^{-1} \left(\frac{\tau^2 S_j}{\tau^2 S_j + \hat{\sigma}_j^2} \hat{\phi}_j + \frac{\hat{\sigma}_j^2}{\tau^2 S_j + \hat{\sigma}_j^2} \mu - \phi_0 \right)^2 + g(S_j, \hat{\sigma}_j, \mu, \tau) \\ &= \left(\frac{S_j}{\hat{\sigma}_j^2 \hat{\beta}_j} \right) (\hat{\phi}_j^* - \phi_0)^2 + g(S_j, \hat{\sigma}_j, \mu, \tau), \end{aligned} \quad (33)$$

where

$$\hat{\phi}_j^* = \hat{\beta}_j \hat{\phi}_j + (1 - \hat{\beta}_j) \mu \quad \text{and} \quad \hat{\beta}_j = \frac{\tau^2 S_j}{\tau^2 S_j + \hat{\sigma}_j^2}. \quad (34)$$

Because $g(\cdot)$ is a term not involving $\hat{\phi}_j$, it should be less relevant to the key parameter ϕ_j of the testing problem. Numerical evidence also suggests that we could ignore the

term, which we will do. This leads to the proposed test, which has a formula similar to the t-test:

$$F_{\text{sm}}^j = t_{jm}^2, \quad \text{where} \quad t_{jm} = (\hat{\phi}_j^* - \phi_0) \left(\frac{S_j}{\hat{\sigma}_j^2 \hat{\beta}_j} \right)^{1/2}. \quad (35)$$

Note that F_{sm}^j uses the estimator $\hat{\phi}_j^*$ which shrinks $\hat{\phi}_j$ toward to μ . In application, the hyper-parameters μ and τ are unknown. Hence in Section 5 we use the data to estimate them in the empirical Bayes fashion.

The formula of (35) works only for $\tau > 0$. Later on if τ is estimated to be zero, (32) is used instead. This principle applies to all the proposed tests of this paper.

Our numerical studies show that F_{sm} is a reasonable approximation of (31) even when the sample size is as small as $T = 10$. Also, the numerical results in Section 7 indicate that F_{sm} has higher average power than the t-test.

4.4 The test shrinking the means and the variances: F_{ss}

To produce a test shrinking both means and variances, we assume as in Case 3 of Section 3 where (σ_j, ϕ_j) follow the prior distribution $\pi(\sigma, \phi) = \pi_1(\sigma)\pi_2(\phi)$, where $\pi_2(\cdot)$ is the normal distribution defined in (26), and $\pi_1(\cdot)$ is the pdf of σ with the distribution of σ^2 being defined right after (23). Applying Theorem 5 and (20) to model (5) and replacing \mathbf{y} by \mathbf{y}_j yields the MAP statistic:

$$\begin{aligned} & \frac{\int \int f(\mathbf{y}_j | \phi, \sigma^2) d\pi_2(\phi) d\pi_1(\sigma)}{\int f(\mathbf{y}_j | \phi_j = \phi_0, \sigma^2) d\pi_1(\sigma)} \\ &= \frac{\int (1/\sigma)^{T-1} e^{-\frac{(T-1)\hat{\sigma}_j^2}{2\sigma^2}} \sqrt{\frac{\sigma^2}{S_j\tau^2 + \sigma^2}} e^{-\frac{1}{2}(\hat{\phi}_j - \mu)^2 \frac{S_j}{S_j\tau^2 + \sigma^2}} d\pi_1(\sigma)}{\int (1/\sigma)^{T-1} e^{-\frac{(T-1)\hat{\sigma}_j^2}{2\sigma^2} - \frac{S_j(\hat{\phi}_j - \phi_0)^2}{2\sigma^2}} d\pi_1(\sigma)}, \end{aligned} \quad (36)$$

where the numerator is derived using similar calculations leading to (30). Assume $\pi_1(\cdot)$ is a log normal distribution with mean μ_V and variance τ_V^2 as we derived F_{sv} . Then we can approximate the MAP test by substituting σ_j^2 by its Bayes estimator $\hat{\sigma}_{jE}^2$ in the numerator and denominator of (36), and obtain an approximation of the MAP test,

$$\sqrt{\frac{\hat{\sigma}_{jE}^2}{S_j\tau^2 + \hat{\sigma}_{jE}^2}} e^{\frac{1}{2}(\hat{\phi}_j - \phi_0)^2 \frac{S_j}{\hat{\sigma}_{jE}^2} - \frac{1}{2}(\hat{\phi}_j - \mu)^2 \frac{S_j}{S_j\tau^2 + \hat{\sigma}_{jE}^2}}. \quad (37)$$

Similar to the calculations leading to (33), we ignore the first multiple term of (37)

and take a log transformation of the reminder, yielding

$$\begin{aligned} & (\hat{\phi} - \phi_0)^2 \frac{S_j}{\hat{\sigma}_{jE}^2} - (\hat{\phi}_j - \mu)^2 \frac{S_j}{S_j \tau^2 + \hat{\sigma}_{jE}^2} \\ &= \left(\frac{S_j}{\hat{\sigma}_{jE}^2 \hat{\beta}_{jE}} \right) (\hat{\phi}_{jE}^* - \phi_0)^2 + g^*(S_j, \hat{\sigma}_j, \mu, \tau), \end{aligned} \quad (38)$$

where $\hat{\beta}_{jE} = \frac{\tau^2 S_j}{\tau^2 S_j + \hat{\sigma}_{jE}^2}$ and $\hat{\phi}_{jE}^* = \hat{\beta}_{jE} \hat{\phi}_j + (1 - \hat{\beta}_{jE}) \mu$. Also $g^*(.)$ is a term involving no $\hat{\phi}_j$. Note we do not need to recalculate (38) again; we simply replace $\hat{\sigma}_j^2$ by $\hat{\sigma}_{jE}^2$ and $\hat{\beta}_j$ by $\hat{\beta}_{jE}$ in (34).

Omitting $g^*(.)$ of (38) yields the proposed test:

$$F_{SS}^j = t_{jEm}^2, \quad \text{where} \quad t_{jEm} = (\hat{\phi}_{jE}^* - \phi_0) \left(\frac{S_j}{\hat{\sigma}_{jE}^2 \hat{\beta}_{jE}} \right)^{1/2}. \quad (39)$$

The expression of F_{SS}^j not only has a compact formula similar to the t-test, but also enjoys nice interpretations. Compared with the t-test, F_{SS}^j uses the shrinkage variance estimator $\hat{\sigma}_{jE}^2$ instead of $\hat{\sigma}_j^2$, and the shrinkage estimator $\hat{\phi}_{jE}^*$ instead of $\hat{\phi}_j$. Therefore, F_{SS}^j shrinks the variances as F_{SV} does and shrinks the means as F_{SM} does. Thus we would expect that F_{SS}^j should perform the best among all the tests. Numerical studies in Section 7 confirm this expectation.

Note that we do not need to assume a large T in deriving (37) whereas we need it to derive (32). Thus F_{SS}^j should be close to the MAP test even for small T .

5 The One-sided Test

We consider the one-sided test: for $1 \leq j \leq N$

$$H_0^j : \quad \phi_j = \phi_0 \quad \text{vs.} \quad H_1^j : \quad \phi_j < \phi_0. \quad (40)$$

The t-test is to reject H_0^j if $t_j = (\hat{\phi}_j - \phi_0) \left(\frac{S_j}{\hat{\sigma}_j^2} \right)^{1/2}$ is smaller than a critical value. To construct tests having a larger average power, we derive RF_{SV} , RF_{SM} and RF_{SV} which shrink the variances, the means and both the variances and means, respectively. We include “R” in the names of these tests since ϕ_0 in the null hypothesis is on the right-hand side of the alternative region.

The test shrinking the variances : RF_{SV}

Suppose ϕ_j is fixed and unknown, and σ_j follows the prior distribution, defined

right after (23), which was used in deriving F_{SV} . Theorems 1-3 can be directly applied to this problem. Using the same arguments leading to F_{SV} , we end up with the test statistic:

$$RF_{SV}^j \equiv t_{jE} = (\hat{\phi}_j - \phi_0) \left(\frac{S_j}{\hat{\sigma}_{jE}^2} \right)^{1/2}. \quad (41)$$

The null hypothesis will be rejected if RF_{SV}^j is smaller than a critical value.

The test shrinking the means: RF_{SM}

Suppose σ_j is fixed and unknown and ϕ_j is a random variable. Since the alternative region is $\phi_j < \phi_0$, we postulate that prior distribution is $N(\mu, \tau^2)$ truncated to the range $(-\infty, \phi_0)$. Hence its pdf is

$$f(\phi|\mu, \tau) = \frac{1}{\Phi(\frac{\phi_0 - \mu}{\tau})} \frac{1}{\sqrt{2\pi}\tau} e^{-\frac{1}{2\tau^2}(\phi - \mu)^2}, \quad \text{when } -\infty < \phi < \phi_0, \quad (42)$$

where $\Phi(\cdot)$ is the cumulative function of a standard normal distribution. By Theorem 4, an approximate MAP test is to reject H_0^j if

$$\frac{\sup_{\sigma_j} \int_{-\infty}^{\phi_0} f(\mathbf{y}_j|\phi, \sigma_j^2) d\pi_2(\phi)}{\sup_{\sigma_j} f(\mathbf{y}_j|\phi_j = \phi_0, \sigma_j^2)} \quad \text{is large.} \quad (43)$$

Similar to the derivation of F_{SM} , a close form for the denominator can be found by replacing $\hat{\sigma}_j^2$ with its maximum point $\hat{\sigma}_{0j}^2$ defined in (28). However, it does not appear that the numerator has a close form and hence we simply replace σ_j^2 with $\hat{\sigma}_j^2$ defined in (4). This leads to

$$\frac{1}{\Phi(\frac{\phi_0 - \mu}{\tau})} \sqrt{\frac{\hat{\sigma}_j^2}{S_j \tau^2 + \hat{\sigma}_j^2}} \Phi(-t_{jm}) (\hat{\sigma}_{0j}^2 / \hat{\sigma}_j^2)^{(T-1)/2} e^{-\frac{1}{2}(\hat{\phi}_j - \mu)^2 \frac{S_j}{S_j \tau^2 + \hat{\sigma}_j^2}}, \quad (44)$$

where t_{jm} is defined in (35). By adopting the arguments in (32) and (33) for deriving F_{SM} , the log transformation of $(\hat{\sigma}_{0j}^2 / \hat{\sigma}_j^2)^{(T-1)/2} e^{-\frac{1}{2}(\hat{\phi}_j - \mu)^2 \frac{S_j}{S_j \tau^2 + \hat{\sigma}_j^2}}$ can be expressed as $(1/2)t_{jm}^2$. Therefore, the MAP test can be approximated, after ignoring the first two terms of (44) and taking the log transformation of the remainder, by

$$\log(\Phi(-t_{jm})) + \frac{t_{jm}^2}{2}. \quad (45)$$

Using the inequality $1 - \Phi(x) < \frac{1}{x}\phi(x)$ for $x > 0$, (45) can be shown to be decreasing in t_{jm} for $t_{jm} > 0$. It is obvious that (45) is also decreasing for $t_{jm} < 0$. Hence (45) is equivalent to the proposed test which rejects H_0^j when

$$RF_{\text{sm}}^j \equiv t_{jm} = (\hat{\phi}_j^* - \phi_0) \left(\frac{S_j}{\hat{\sigma}_j^2 \hat{\beta}_j} \right)^{1/2} \text{ is small.} \quad (46)$$

The test shrinking the variances and the means: RF_{ss}

Under the assumption of Case 3 in Section 3, suppose (σ, ϕ) follows a prior distribution $\pi(\sigma, \phi) = \pi_1(\sigma)\pi_2(\phi)$, where $\pi_1(\cdot)$ is the pdf of σ with the distribution of σ^2 being defined right after (23) and $\pi_2(\cdot)$ is the truncated normal distribution defined in (42). Then the MAP test is

$$\frac{\int \int_{-\infty}^{\phi_0} f(\mathbf{y}_j | \phi, \sigma^2) d\pi_2(\phi) d\pi_1(\sigma)}{\int f(\mathbf{y}_j | \phi_j = \phi_0, \sigma^2) d\pi_1(\sigma)}. \quad (47)$$

Instead of integrating with respect to σ , we replace σ_j in the numerator and denominator by $\hat{\sigma}_{jE}$. An approximation of the MAP test is obtained,

$$\frac{1}{\Phi(\frac{\phi_0 - \mu}{\tau})} \sqrt{\frac{\hat{\sigma}_{jE}^2}{S_j \tau^2 + \hat{\sigma}_{jE}^2}} \Phi(-t_{jEm}) e^{\frac{1}{2}(\hat{\phi} - \phi_0)^2 \frac{S_j}{\hat{\sigma}_{jE}^2} - \frac{1}{2}(\hat{\phi}_j - \mu)^2 \frac{S_j}{S_j \tau^2 + \hat{\sigma}_{jE}^2}}, \quad (48)$$

where t_{jEm} is defined in (39). Ignoring the first two terms of (48), taking the log transformation of the remainder, and using the leading term in (38) to substitute for the exponent yield the proposed statistic:

$$RF_{\text{ss}}^j \equiv t_{jEm} = (\hat{\phi}_j^* - \phi_0) \left(\frac{S_j}{\hat{\sigma}_{jE}^2 \hat{\beta}_{jE}} \right)^{1/2}. \quad (49)$$

6 Estimating the Hyper-parameters and the Critical Values

6.1 Estimate the hyper-parameters: μ and τ^2

The two-sided test: Normal distribution

We follow the empirical Bayes approach and use data to estimate the hyper-parameters (μ, τ^2) used in F_{sm} and F_{ss} . Suppose ϕ_j follows $N(\mu, \tau^2)$ with probability θ_1 , and $\phi_j = \phi_0$ with probability $\theta_0 = 1 - \theta_1$. Note that θ_1 can be interpreted as N_1/N .

By assuming the independence of \mathbf{y}_j for $j = 1, \dots, N$, the log likelihood function of (μ, τ^2) is

$$\begin{aligned}
& \log(f(\mathbf{y}_1, \dots, \mathbf{y}_N | \mu, \tau^2)) \\
&= \sum_{j=1}^N \log(f(\mathbf{y}_j | \mu, \tau^2)) \\
&= \sum_{j=1}^N \log\{\theta_1 \int f(\mathbf{y}_j | \mu, \tau^2, \phi_j = \phi) d\pi_2(\phi) + (1 - \theta_1) f(\mathbf{y}_j | \phi_j = \phi_0)\} \quad (50) \\
&= C + \sum_{j=1}^N \log\left\{\theta_1 \sqrt{\frac{\sigma_j^2}{S_j \tau^2 + \sigma_j^2}} e^{-\frac{1}{2} \frac{S_j}{S_j \tau^2 + \sigma_j^2} (\hat{\phi}_j - \mu)^2} + (1 - \theta_1) e^{-\frac{S_j}{2\sigma_j^2} (\hat{\phi}_j - \phi_0)^2}\right\},
\end{aligned}$$

where C is a constant not depending on μ and τ^2 . One can maximize (50) and derive the maximum likelihood estimator for (μ, τ^2, θ_1) . However, this involves the maximization of three variables. Instead, we propose an estimator which is easier to compute. We use the approximation $\hat{\phi}_j | \phi_j, \sigma_j \sim N(\phi_j, \frac{\sigma_j^2}{S_j})$. Hence

$$E(\hat{\phi}_j | \phi_j, \sigma_j) \doteq \phi_j \quad \text{and} \quad E(\hat{\phi}_j^2 | \phi_j, \sigma_j) \doteq \phi_j^2 + \frac{\sigma_j^2}{S_j}. \quad (51)$$

Therefore,

$$\begin{aligned}
E(\hat{\phi}_j) &= \theta_1 E(\phi_j | \mu, \tau) + (1 - \theta_1) \phi_0 = \theta_1 \mu + (1 - \theta_1) \phi_0 \\
E(\hat{\phi}_j^2) &= \theta_1 E(\phi_j^2 + \frac{\sigma_j^2}{S_j} | \mu, \tau) + (1 - \theta_1) (\phi_0^2 + E(\frac{\sigma_j^2}{S_j})) \quad (52) \\
&= E(\frac{\sigma_j^2}{S_j}) + \theta_1 (\mu^2 + \tau^2) + (1 - \theta_1) \phi_0^2.
\end{aligned}$$

Let m_1 and m_2 denote $E(\hat{\phi}_j)$ and $E(\hat{\phi}_j^2)$, respectively. Solving μ and τ^2 in terms of m_1 , m_2 and θ yields

$$\mu = \frac{m_1 - (1 - \theta_1) \phi_0}{\theta_1} \quad \text{and} \quad \tau^2 = \frac{m_2 - (1 - \theta_1) \phi_0^2 - E(\frac{\sigma_j^2}{S_j})}{\theta_1} - \mu^2. \quad (53)$$

Substitute m_1 and m_2 with $\hat{m}_1 = (1/N) \sum_{j=1}^N \hat{\phi}_j$ and $\hat{m}_2 = (1/N) \sum_{j=1}^N \hat{\phi}_j^2$. Furthermore, replace $E(\frac{\sigma_j^2}{S_j})$ in (53) with $(1/N) \sum_{j=1}^N \sigma_j^2 / S_j$, where σ_j^2 is, in turn, replaced with $\hat{\sigma}_{jE}^2$ for F_{ss} (and $\hat{\sigma}_j^2$ for F_{sm}). The latter substitution for σ_j^2 is also applied to (50). Moreover, plug the resultant formula for μ and τ^2 into (50). Then the resultant pseudo likelihood function is a function of θ_1 only. We then estimate θ_1 by $\hat{\theta}_1$ which maximizes the function. Using $\hat{\theta}_1$, \hat{m}_1 and \hat{m}_2 , we may estimate μ and τ^2 based on (53).

The one-sided test: Truncated normal distribution

To estimate the hyper-parameter (μ, τ^2) of a truncated normal distribution used

to derive RF_{Sm} or RF_{ss} , we adopt the empirical Bayes approach again, assuming that ϕ_j follows the truncated normal distribution with probability θ_1 and $\phi_j = \phi_0$ with probability $1 - \theta_1$. Therefore, the log likelihood function of (μ, τ^2) is

$$\begin{aligned} & \log(f(\mathbf{y}_1, \dots, \mathbf{y}_N | \mu, \tau^2)) \\ &= C + \sum_{j=1}^N \log \left\{ \theta_1 \frac{1}{\Phi(\frac{\phi_0 - \mu}{\tau})} \Phi(-t_{jm}^o) \sqrt{\frac{\sigma_j^2}{S_j \tau^2 + \sigma_j^2}} e^{-\frac{1}{2} \frac{S_j}{S_j \tau^2 + \sigma_j^2} (\hat{\phi}_j - \mu)^2} + (1 - \theta_1) e^{-\frac{S_j}{2\sigma_j^2} (\hat{\phi}_j - \phi_0)^2} \right\}, \end{aligned} \quad (54)$$

where t_{jm}^o is identical to t_{jm} in but using σ_j to replace $\hat{\sigma}_j$.

Using (51) and the moments of a truncated normal distribution, we have the following results after some calculations:

$$\begin{aligned} E(\hat{\phi}_j) &= \theta_1 (\mu - \lambda(\alpha)\tau) + (1 - \theta_1)\phi_0 \\ E(\hat{\phi}_j^2) &= E(\frac{\sigma_j^2}{S_j}) + \theta_1 \{ \tau^2 (1 - \alpha\delta(\alpha)) + (\mu - \lambda(\alpha)\tau)^2 \} + (1 - \theta_1)\phi_0^2 \end{aligned} \quad (55)$$

where $\alpha = \frac{\phi_0 - \mu}{\tau}$, $\lambda(\alpha) = \frac{\phi(\alpha)}{\Phi(\alpha)}$ and $\delta(\alpha) = \lambda(\alpha)(\alpha + \lambda(\alpha))$. Replacing $E(\hat{\phi}_j)$ and $E(\hat{\phi}_j^2)$ by m_1 and m_2 in (55), respectively, gives us

$$\begin{aligned} \mu &= \frac{m_1 - (1 - \theta_1)\phi_0}{\theta_1} + \lambda(\alpha)\tau \\ \tau^2 &= \left\{ \frac{m_2 - (1 - \theta_1)\phi_0^2 - E(\frac{\sigma_j^2}{S_j})}{\theta_1} - (\mu - \lambda(\alpha)\tau)^2 \right\} / (1 - \alpha\delta(\alpha)). \end{aligned} \quad (56)$$

Since the right-hand side of (56) still involves μ and τ^2 , an iteration algorithm is proposed to estimate (μ, τ^2) . We use the estimator for (μ, τ^2) in the two-sided case depicted above as the initial value to obtain a function of θ_1 only. Calculate $\hat{\theta}_1$ that maximizes the function. Now plug $\hat{\theta}_1$ and the initial value of (μ, τ^2) into the right-hand side of (56) to obtain a new estimator of μ and τ^2 . The process is repeated to obtain a new estimator of θ_1 , μ and τ^2 . In the above calculation m_1 and m_2 are replaced by $(1/N) \sum_{j=1}^N \hat{\phi}_j$ and $(1/N) \sum_{j=1}^N \hat{\phi}_j^2$, respectively. Also $E(\frac{\sigma_j^2}{S_j})$ is replaced by $(1/N) \sum_{j=1}^N \sigma_j^2 / S_j$, where σ_j^2 is, in turn, replaced by $\hat{\sigma}_{jE}^2$ for F_{ss} (and $\hat{\sigma}_j^2$ for F_{Sm}). The latter substitutions for σ_j^2 are also applied to (54).

6.2 Generating the critical value by the Bootstrap method

In order to have a good finite sample property, we should use the bootstrap method to determine the critical values of the proposed tests. In what follows, we present the

details of the bootstrap procedure for the two-sided test. A similar procedure can be applied to the one-sided test.

Let

$$\hat{e}_{j,t} = y_{j,t} - \hat{\phi}_j y_{j,t-1} \quad \text{for } 2 \leq t \leq T \text{ and } 1 \leq j \leq N. \quad (57)$$

Under the null hypothesis, we use the hypothesized value ϕ_0 to create the following bootstrap sample for the j -th group, $\{y_{j,t}^*, t = 1, 2, \dots, T\}$, where

$$y_{j,t}^* = \phi_0 y_{j,t-1}^* + e_{j,t}^*, \quad t = 1, 2, \dots, T, \quad (58)$$

and $e_{j,t}^*$'s are sampled with replacement from $\{\hat{e}_{j,t}, 2 \leq t \leq T\}$.

One can plug $y_{j,t}^*$'s into the t-statistic, F_{SV} statistic, F_{SM} statistic and F_{SS} statistic. For each statistic, repeat it R times and calculate the percentile (95%tile for 5% test) which is then used as the critical value.

Note that in calculating the critical values for F_{SM} and F_{SS} , we use data to estimate μ and τ once and from then on, μ and τ are set to be identical to its estimated value. Hence in each bootstrap sample, μ and τ are not re-estimated. This is reasonable since in the bootstrap sample, ϕ_j is taken to be ϕ_0 , the hypothesized value. The bootstrap samples do not have information about ϕ_j and hence they should not be used to estimate the hyper-parameters of ϕ_j . Regarding $\hat{\sigma}_{jE}^2$ used in the two tests F_{SV} and F_{SS} , we do recalculate its value for each bootstrap sample, since they contain the information about σ_j^2 .

7 Simulation Studies

7.1 The white noise hypothesis: Two-sided test for $\phi_0 = 0$

This simulation considers a special case of the two-sided test in which the null and the alternative hypotheses are, respectively, $H_0^j: \phi_j = 0$ and $H_1^j: \phi_j \neq 0$ for $1 \leq j \leq N$. The null hypothesis is commonly referred to as the white noise hypothesis.

In Section 6, we estimate the hyper-parameters of the proposed tests under the assumption that cross section series are mutually independent; namely, the N series are independent. However, the following simulation studies both the independent and dependent cases. In general, we assume a multi-factor error structure (Pesaran et al., 2013), which includes both independent and dependent cases, in order to check the robustness of the proposed tests with respect to cross section dependence. It

turns out that in both the independent case and dependent case, the proposed tests improve uniformly over the t-test. Hence in both cases, the proposed tests apparently “borrow the strength” from all the populations to do better than the t-test.

Specifically, the data are generated using the model

$$y_{j,t} = \phi_j y_{j,t-1} + e_{j,t} \quad \text{for } 1 \leq j \leq N, \text{ with } \phi_j = 0 \text{ for } j > N_1, \quad (59)$$

$$e_{j,t} = c_{j,1} f_{1,t} + c_{j,2} f_{2,t} + \epsilon_{j,t}, \quad (60)$$

where $\epsilon_{j,t}$, $1 \leq j \leq N$, $1 \leq t \leq T$, are independently $N(0, \sigma_j^2)$ distributed.

Model (60) is called a multi-factor model, which reduces to the independent model when $c_{j,1} = c_{j,2} = 0$. Otherwise, $\{e_{j,t}\}$, for $1 \leq j \leq N$, are dependent. For the dependent case studied below, $c_{j,1}$ and $c_{j,2}$ are generated as random samples from the uniform distribution over $[0, 1]$ and $[0, 2]$ respectively.

For t-statistic, F_{SV} , F_{SM} and F_{SS} , we then calculate the average power ETP/N_1 and the average type one error $EFP/(N - N_1)$, where ETP and EFP are defined in (8) and (9).

The parameters σ_j , $1 \leq j \leq N$, are i.i.d samples generated from $\pi_1(\sigma)$ and ϕ_j , $1 \leq j \leq N_1$, are i.i.d samples generated from $\pi_2(\phi)$, where π_1 and π_2 will be specified below.

Normal prior distributions

The prior distributions assumed are

$$\begin{aligned} \pi_1(\sigma) : \quad \ln(\sigma_j^2) &\sim N(\mu_V, \tau_V^2) \quad \text{for } 1 \leq j \leq N, \\ \pi_2(\phi) : \quad \phi_j &\sim N(\mu, \tau^2) \quad \text{for } 1 \leq j \leq N_1. \end{aligned} \quad (61)$$

We now examine the average power and the average type one error of the t-test and our proposed tests. In each of Figures 1.1 through 1.6, the simulated average power and the average type one error are plotted, against μ , in solid curve and dotted curve respectively. In the simulation, each point is based on averaging at least 4,000 replications.

In Figures 1.1 through 1.5, the cross section series are mutually independent. For various settings of T , N and N_1 , τ and the coefficient of variation ($CV =$

τ_V/μ_V) specified in the headings of these figures, the figures basically show that

statement (i) : all the proposed tests have uniformly higher average power than the t-test,

statement (ii) : the uniformly most powerful test is F_{SS} test.

(62)

And, the average power of F_{SS} could be 70%, as shown in Figures 1.1, larger than that of the t-test.

Moreover all the tests have average type one error controlled under 5% level with the exception of Figure 1.5, which correspond to small N and N_1 . Further numerical study shows that the discrepancy is due to the estimation error of μ and τ , which is larger for small N and N_1 . However, even in Figure 1.5, the average type one errors of alternative tests are only slightly larger than 0.05.

As for the case of cross section dependence, we adopt the multi-factor model (60) to generate the data. Under the same settings of T , N and N_1 , τ and CV as those in Figures 1.1 through 1.5, we obtain very similar graphs showing basically that the statements (i) and (ii) in (62) hold. We only report Figure 1.6 having the setting of Figure 1.1.

In fact, the study shows that the improvements of F_{SS} test over the t-test are slightly larger in some of the dependent cases. This is intuitively reasonable since a procedure shrinking toward the common means or variances should be expected to do better when the sections are more correlated.

Our simulation studies also confirm the effectiveness of the estimator for the hyper-parameter (μ, τ^2) in Section 6.1. More specifically, the average power of the proposed tests using the estimated (μ, τ^2) is very similar to that of the tests using the true values, although the average power corresponding to the true values is not reported here.

In Econometrics, it is important to focus on the alternative hypothesis which is close to the null hypothesis. This is especially true for the unit root test, to be discussed in Section 7.2. Consequently, the tests do not have large average power. However, the increase of the average power by 0.05 will, on the average, increase the detected true positives by $(0.05)N$, which could be quite substantial when N is large.

Uniform prior distributions and fixed effect model

To show the robustness of the proposed tests with respect to the miss-specification of prior distributions, we use “wrong” distributions such as the uniform distributions and fixed effect model to generate parameters. We consider the uniform distributions as

$$\begin{aligned}\pi_1(\sigma) : \quad & \ln(\sigma_j^2) \sim U(2 - 2\sqrt{3}\tau_V, 2 + 2\sqrt{3}\tau_V) \quad \text{for } 1 \leq j \leq N, \\ \pi_2(\phi) : \quad & \phi_j \sim U(\mu - 2\tau, \mu + 2\tau) \quad \text{for } 1 \leq j \leq N_1.\end{aligned}\tag{63}$$

We write the distribution of $\ln(\sigma_j^2)$ this way, so that the variance is $4\tau_V^2$ and the mean is two; consequently $CV = \tau_V$. For such a prior, we plot the average power using the same settings as Figures 1.1 through 1.5 for both independent case and multi-factor models. The resultant graphs are similar to Figures 1.1 through 1.5. We report only Figure 2.1 (corresponding to the independent case) and Figure 2.2 (corresponding to the multi-factor model) both having the same settings as Figures 1.1. Both figures and the unreported figures basically confirm the two statements in (62).

To study the fixed effect model, i.e. ϕ_j being fixed, let

$$\begin{aligned}\phi_j &= \mu - 2\tau \quad \text{for } 1 \leq j \leq N_1/2, \\ \phi_j &= \mu + 2\tau \quad \text{for } N_1/2 + 1 \leq j \leq N_1,\end{aligned}\tag{64}$$

and $\sigma_j = \sigma$ for all j ($CV=0$). The results displayed in Figures 2.3 and 2.4 show that the improvements obtained by the proposed tests are also robust with respect to this “wrong” setting. Statements in (62) basically hold.

Conditional heteroscedasticity

Below, we shall generate $\epsilon_{j,t}$ from a GARCH(1,1) model instead of an i.i.d Normal model. The GARCH(1,1) model is commonly used in Finance and Economics to describe conditionally heteroscedastic phenomena. The GARCH(1,1) model used is

$$\epsilon_{j,t} = \omega_{j,t} \epsilon_{j,t}^* \quad \text{where} \quad \omega_{j,t}^2 = 1 + 0.8\omega_{j,t-1}^2 + 0.15\epsilon_{j,t-1}^2;\tag{65}$$

where $\epsilon_{j,t}^*$ are i.i.d. standard normal random variables and $\omega_{j,t}^2$ is the conditional variance of $\epsilon_{j,t}$. Models (59), (60) and (61) are still assumed except $\epsilon_{j,t}$ follows (65). The results in Figures 3.1 and 3.2 show that the proposed tests

still improve on the t-test. In particular, Figure 3.2 assumes a model that has cross section dependence and conditional heteroscedasticity. Therefore the improvements are quite robust with respect to dependence and heteroscedasticity. Statements in (62) are basically correct.

Large dimensional series

In what follows, we study the large dimensional series that $N = 1000$ and $N_1 = 500$. Under the settings of Figure 1.1, Figures 4.1 and 4.2 report the results of independent and dependent cases, respectively. Both figures strongly confirm the two statements in (62) and show that the improvements provided by the proposed tests, including F_{SV} , F_{SM} and F_{SS} , over the t-test increase slightly when the dimension increases.

7.2 The unit root hypothesis: One-sided test for $\phi_0 = 1$

Now we apply all the tests to the unit root hypothesis, for testing $H_0^j: \phi_j = 1$ vs. $H_1^j: \phi_j < 1$ for $1 \leq j \leq N$.

We generate the data using the model

$$y_{j,t} = \phi_j y_{j,t-1} + e_{j,t} \quad \text{for } 1 \leq j \leq N, \text{ with } \phi_j = 1 \text{ for } j > N_1, \quad (66)$$

where $e_{j,t}$'s are generated from equation (60), $\epsilon_{j,t}$'s from a normal distribution $N(0, \sigma_j^2)$, σ_j^2 's from a prior distribution $\pi_1(\sigma)$ for $1 \leq j \leq N$, and ϕ_j from $\pi_2(\phi)$ for $1 \leq j \leq N_1$, where π_1 and π_2 are specified below.

We shall graph the average powers and the average type one errors of t-test, RF_{SV} and RF_{SS} . However, we do not show the results of RF_{SM} since its performance is very similar to (but slightly worse than) the t-test.

Truncated normal prior distributions

We generate parameters by the prior distributions to derive the proposed tests. That is, $\ln(\sigma_j^2)$ for $1 \leq j \leq N$ have $N(\mu_V, \tau_V^2)$ distribution, and ϕ_j , $1 \leq j \leq N_1$, follow a $N(\mu, \tau^2)$ distribution truncated to the range $(-\infty, 1)$.

In Figures 5.1 through 5.5, we graph the simulated average power (plotted by solid lines), and the simulated average type one error (plotted by dotted lines) of the three tests under the various combinations of T , N , N_1 , CV and τ specified in the headings. These figures deal with the cases when the cross sections are

independent, namely model (60) with $c_{j,1} = c_{j,2} = 0$ for all j . These graphs demonstrate that the following statement (iii) holds:

statement (iii) : RF_{SS} and RF_{SV} basically have uniformly higher average power than the t-test. (67)

Hence statement (i) in (62) basically holds with RF_{SV} and RF_{SS} . Regarding the question as to which test of the two is better, the answer is not clear. In principle, the test RF_{SS} should perform better since it has more to do with the specifics of the priors. However, RF_{SS} is not always the winner. This may have to do with the fact that more hyper-parameters need to be estimated in constructing RF_{SS} than those in constructing RF_{SV} . Estimation of the hyper-parameters is a difficult problem in the one-sided case because of truncation of the prior. This may explain why RF_{SS} is not always the winner.

For the dependent case, we also produce results similar to Figures 5.1 to 5.5. However, only Figure 5.6 is reported which has the same settings as in Figure 5.1. Figure 5.6, for the dependent multi-factor model, is very similar to Figure 5.1 for the independent model. This demonstrates that the improvements of the proposed tests over the t-test are quite robust with respect to the cross section dependence.

Whether one uses RF_{SV} or RF_{SS} , these figures show that both have higher average power than the t-test. The average power of RF_{SS} could be about 25% larger than the t-test (Figures 5.1 and 5.6). The average type one error of all the tests are controlled under or nearly under the 5% level.

Uniform prior distributions and fixed effect model

Below we shall study different priors and models. In all cases, statement (iii) in (67) is shown to be true. Specifically, RF_{SV} and RF_{SS} have uniformly greater power than t-test and there is no clear winner between RF_{SV} and RF_{SS} .

To study how improvements are affected by a "wrong prior", we consider (63) except that ϕ_j is truncated so ϕ_j is in $(-\infty, 1)$ for $1 \leq j \leq N_1$. Following the settings of Figures 5.1 through 5.5, we plot the average powers which show that the fact that the prior is the "wrong" prior has little effect. The resultant figures are very similar. We only report Figure 6.1 (similar to Figure 5.1) and Figure 6.2 (similar to Figure 5.6).

Similar plottings were carried out for a fixed effect model where $\phi_j = \mu - 2\tau$ for $1 \leq j \leq N_1/2$ and $\phi_j = \min(0.99, \mu + 2\tau)$ for $N_1/2 + 1 \leq j \leq N_1$. Since in the graphs, $\mu \leq 1$, the choice of ϕ_j above ensures that $\phi_j < 1$, for $1 \leq j \leq N_1$. Hence the first N_1 hypotheses are the alternative hypotheses. The rest of the hypotheses are the null hypotheses, where $\phi_j = 1$ for $j > N_1$.

We report the results in Figures 6.3 and 6.4 which have the same settings as Figures 6.1 and 6.2, respectively. These two sets of graphs are very similar, confirming statement (iii).

Conditional heteroscedasticity

Figures 7.1 and 7.2 graph the average powers and average type one errors when the data are generated by equation (64) with the GARCH(1,1) error (65), and the parameters are generated by the truncated normal prior distributions. The results confirm statement (iii) and show that the proposed tests still improve on the t-test even when conditional heteroscedasticity and cross section dependence are present.

Large dimensional series

Figures 8.1 and 8.2 demonstrate results when the dimensions $N = 800$ and $N_1 = 600$. The figures show that statement (iii) still holds when dimension is large.

8 Concluding Remarks

To analyze the coefficients of a panel AR(1) model, we propose tests which determine which individual hypothesis should be accepted or rejected. Furthermore, our proposed tests improve on the t-test under the criterion of average power. We derive them using empirical Bayes approach and then using approximation to obtain our proposed tests, which have a form similar to the t-test. The only difference is that, in our proposed tests, the estimators of the means and variances are replaced by shrinkage estimators. The proposed tests “borrow the strength” from all the series to test against every individual series, resulting in more power. Simulation studies show that the proposed tests have significant improvements over the t-test, especially when the sample size T is small and the dimension N is moderate or large. Compared to the

t-test, the average power of F_{SS} and RF_{SS} could be 70% higher in the two-sided test, and 25% higher in the one-sided test respectively.

In this paper, we derive the tests under the assumption that the series are independent; and show that “borrowing the strength” from independent series will improve average power of the t-test. However, simulation demonstrates that the improvement is robust with respect to the cross section dependence. This is only reasonable. A procedure that can do well by “borrowing the strength” even from independent series can certainly do so from dependent series. In this paper, we only work with AR(1) model; we, however, anticipate that these results can be generalized to the other more complex time series models. Since the proposed tests can determine acceptance or rejection of an individual hypothesis, this should prove to be a very useful method in practice.

Appendix

Proof of Theorem 1: The difference in (11) equals

$$\frac{1}{N_1} \sum_{\{j|\phi_j \in D\}} [g_j(\sigma_j) - E(g_j(\sigma_j))], \quad (68)$$

where $g_j(\sigma_j) = P_{\phi_j, \sigma_j}(\mathbf{y}_j \in C) = \int_{\mathbf{y} \in C} f(\mathbf{y}|\phi_j, \sigma_j) d\mathbf{y}$ and $E(g_j(\sigma_j)) = \int g_j(\sigma) d\pi_1(\sigma)$. Since variance of $g_j(\sigma_j) \leq E(g_j^2(\sigma_j)) \leq 1$, the variance of (68) is bounded above by $p_1/p_1^2 = 1/p_1$, which converges to zero. Hence (68) converges in probability to zero by the law of large numbers, completing the proof of (11).

Equation (12) can be proved similar, except noting that $\int \int_{\mathbf{y} \in C} f(\mathbf{y}|\phi_0, \sigma_j) d\mathbf{y} d\pi_1(\sigma_j)$ does not depend on j .

Proof of Theorem 3: It sufficient to show that

$$\sup_{\phi_1, \dots, \phi_N \in D} \frac{1}{N_1} \sum_{\{j|\phi_j \in D\}} \int f(\mathbf{y}|\phi_j, \sigma) d\pi_1(\sigma) = \int f(\mathbf{y}|\hat{\phi}_M, \sigma) d\pi_1(\sigma). \quad (69)$$

Note that the left-hand side is obviously bounded by the right-hand side since $f(\mathbf{y}|\phi_j, \sigma) \leq f(\mathbf{y}|\hat{\phi}_M, \sigma)$ for $\phi_j \in D$. Also replacing ϕ_j by $\hat{\phi}_M$ on the left-hand side leads to a lower bound, which is exactly the right-hand side, establishing (69) and the theorem.

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